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Quantization of Poisson pairs: the R-matrix approach

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We suggest an approach to the quantization problem of two compatible Poisson brackets in the case when one of them is associated with a solution of the classical Yang–Baxter equation. We show that the quantization scheme (a Poisson bracket → an associate algebra, quantizing this bracket in the spirit of the Berezin–Lichnerovicz deformation quantization → its representation in a Hilbert space) has to be enlarged. We represent the deformation algebras, quantizing the "R-matrix" brackets, in a space with an S-symmetric pairing, where S is a solution of the corresponding quantum Yang–Baxter equation. An example of quantization of an "exotic" harmonic oscillator is discussed.

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An R-matrix is an invertible operator $R : V^{\otimes 2} \to V^{\otimes 2}$ satisfying the quantum Yang–Baxter equation (QYBE)

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} .$$

Here $R^{12} = R \otimes \mathrm{id}$ etc. and V is a finite- (or infinite-)dimensional linear space over the field $k = \mathbb{R}$ or $\mathbb{C}$. If $R = \sigma \cdot S$, where σ is the permutation $\sigma(v_1 \otimes v_2) = v_2 \otimes v_1$, $v_i \in V$, then QYBE can be rewritten as follows:

$$(S \otimes \mathrm{id})(\mathrm{id} \otimes S)(S \otimes \mathrm{id}) = (\mathrm{id} \otimes S)(S \otimes \mathrm{id})(\mathrm{id} \otimes S) . \tag{0.1}$$

Thus, an R-matrix in the form (0.1) gives a "local" representation of the braid group. Henceforth we call an operator S satisfying (0.1) a YB operator. And we call a YB operator a symmetry if it satisfies the "unitarity" condition $S^2 = \mathrm{id}$ (or $R^{12} \cdot R^{21} = \mathrm{id}$, where $R^{21} = \sigma R^{12} \sigma$ for R).

A classical analogue of this object is a "classical R-matrix", i.e. an operator $R : V^{\otimes 2} \to V^{\otimes 2}$ satisfying the equation

$$\langle R,R\rangle=[R^{12},R^{13}]+[R^{12},R^{23}]+[R^{13},R^{23}]=0\,. \tag{0.2}$$

The unitarity condition for this operator has the form $R^{12}+R^{21}=0$. R is usually assumed to be an element of $\Lambda^2\mathfrak{G}$ where $\mathfrak{G}$ is a Lie algebra and (0.2) is interpreted as a relation in $(\mathfrak{G}\oplus k\cdot 1)^{\otimes 3}$. For a given representation φ of the Lie algebra $\mathfrak{G}$ the operator $(\varphi\otimes\varphi)R$ is a classical R-matrix in the sense described above.

We fix a Lie algebra $\mathfrak{G}$ and interpret R as an element of $\Lambda^2\mathfrak{G}$. Consider a representation φ of the Lie algebra $\mathfrak{G}$ in the space $\mathrm{Der}(M)$ of smooth vector fields on a smooth manifold M and a bilinear operator

$$f\otimes g\to\{f,g\}=\mu(\varphi\otimes\varphi)R\rfloor(\mathrm{d}f\otimes\mathrm{d}g)\,.$$

Here and below μ denotes the "usual" multiplication. Since $(\varphi\otimes\varphi\otimes\varphi)\langle R,R\rangle$ is the Schouten bracket of the bivector $(\varphi\otimes\varphi)R$ with itself, (0.2) provides a sufficient (but not necessary) condition for this bilinear operation to define a Poisson bracket.

If G is a Lie group, $\mathfrak{G}$ is the corresponding Lie algebra and φ is the representation of $\mathfrak{G}$ by left- (or right-)invariant fields on G, then we get a left- (or right-) invariant Poisson bracket (defined in ref. [1]), and if φ is the adjoint representation on G or $\mathfrak{G}^*$ then we get another pair of brackets introduced in ref. [2] (the bracket on $\mathfrak{G}^*$ was independently introduced by Magry).

We do not consider here the Hamilton–Lie bracket (in the terminology of ref. [1]), which is the difference between the above-mentioned left- and right-invariant brackets and which can be generalized to the case when the tensor $\langle R,R\rangle$ is G-invariant. We would only like to stress the fact that quantization of this bracket leads to the so-called quantum groups.

Let $\{\ ,\ \}$ be a Poisson bracket on a smooth manifold M and φ be a representation of the Lie algebra $\mathfrak{G}$ in the space $\mathrm{Ham}(M,\{\ ,\ \})$ of Hamiltonian vector fields. Then the Poisson brackets $\{\ ,\ \}_R$ and $\{\ ,\ \}$ form a Poisson pair, i.e., any linear combination $\{\ ,\ \}_{a,b}=a\{\ ,\ \}+b\{\ ,\ \}_R$ is a Poisson bracket.

In this paper we construct a simultaneous quantization of this family of Poisson brackets. It should be emphasized that we consider the quantization of a Poisson bracket $\{\ ,\ \}$ as a deformation algebra equipped with an associative multiplication

$$f\otimes g\to f*_\hbar g,\quad f,g\in C^\infty(M)\,,$$

depending on the parameter $\hbar$ and satisfying the "correspondence principle" together with a representation of this algebra in a linear space V.

We show that, if the algebra quantizing the initial bracket $\{\ ,\ \}$ is represented by self-adjoint operators in a Hilbert space (substituting $i\hbar$ instead of $\hbar$ in the correspondence principle), then it turns out to be natural to represent the algebras quantizing the other brackets $\{\ ,\ \}_{a,b}$ in the same linear space but equipped with deformed (non-symmetric) pairings. Consider the symplectic two-form Ω

on M. Let $L_2(M, \Omega^n)$ ($n = \dim M/2$) be the Hilbert space where the quantizing algebra of the initial Poisson brackets is represented. Then the linear term of the deformation of the Hilbert pairing is $\int \{f, g\}_R \Omega^n$ (up to a factor). Here and below we assume that all functions used in integrations decrease rapidly towards infinity and all integrals are well defined.

Note that the functional $f \to \int f \Omega^n$ is a central functional on the Poisson algebra with the bracket $\{\ ,\ \}$, i.e., $\int \{f, g\} \Omega^n = 0$ (for any f and g). For degenerate brackets (the brackets $\{\ ,\ \}_{a,b}$ are usually degenerate) central functionals do not exist. So there are no central functionals on the algebras quantizing these brackets. We construct an S-trace and an S-symmetric pairing on these algebras. Here $S : V^{\otimes 2} \to V^{\otimes 2}$ is the symmetry quantizing the initial classical R-matrix R. These objects are the results of deformation of the initial trace and corresponding pairing.

Note that an S-trace arises naturally in the theory of monoidal quasi-tensor categories (see ref. [3]). Here we confine ourselves to some special objects of such a category only, so we do not use the general category framework. A more general approach includes quasi-tensor categories with non-trivial associativity morphisms. But it should be emphasized that such categories are necessary to apply our scheme to quantization of some other brackets. This situation will be studied elsewhere.

The conjugation operator in the operator space is also deformed together with the deformation of pairing. The conjugation $A \to A^*$ constructed below is an involutive operator which does not satisfy the condition $(A \cdot B)^* = B^* \cdot A^*$ but it satisfies an analogue of this condition (see section 3). This is a crucial point of our construction. In order to quantize some degenerate Poisson brackets (namely the R-matrix Poisson brackets) we need to enlarge the quantization scheme and to generalize the trace, the pairing and the conjugation operator to the S-trace, the S-pairing and the S-conjugation, respectively.

This paper is organized as follows. In the first two sections we construct the R-matrix Poisson bracket and the deformation algebras quantizing the brackets $\{\ ,\ \}_{a,b}$. In section 3 we present a scheme of "S-quantum mechanics".

It should be noted that this scheme includes not only "quasiclassical" symmetries S (arising in the quantization of classical R-matrices) but also general YB symmetries (the existence of non-quasiclassical symmetries was shown in ref. [4]). In section 4 we show that the quasiclassical objects constructed in section 2 satisfy the requirements of the scheme proposed in section 3.

The final section 5 contains an important example – an "S-quantum harmonic oscillator". Here we do not confine ourselves to the framework of the quasiclassical situation. The main purpose of this example is to demonstrate that the statistical sum of the energy operator depends on the Poincaré series connected with the symmetry S. Note that the relevant statistics differs from both the Bose–Einstein and the Fermi–Dirac ones.

We regard this paper to be an initial step of a wide investigation program. As the second step we will introduce new dynamic models with non-local interactions and quantize them. We also hope to obtain a new tool for studying the problem of quantum anomalies.

1. Poisson brackets associated with R-matrices

Consider a Poisson bracket in the space $C^\infty(M)$,

$$f\otimes g\to\{f,g\}\in C^\infty(M),\quad f,g\in C^\infty(M)\,,$$

and let the Poisson action of a connected and simply connected Lie group G be defined on a smooth manifold M. We obtain the natural representation $\varphi:\mathfrak{G}\to\mathrm{Der}(M,\{\ \})$. Here $\mathfrak{G}$ is the Lie algebra of the group G, $\mathrm{Der}(M,\{\ \})$ is the Lie algebra of all Poisson vector fields on M.

Consider a classical R-matrix $R\in\Lambda^2\mathfrak{G}$, $R=r^{ij}X_i\otimes X_j$, $r^{ij}\in\mathbb{R}$, where $\{X_i\}$ is a fixed basis of $\mathfrak{G}$.

Consider the bilinear operator from $C^\infty(M)^{\otimes 2}$ to $C^\infty(M)$,

$$f\otimes g\to\{f,g\}_R=\mu(\varphi\otimes\varphi)R\rfloor(\mathrm{d}f\otimes\mathrm{d}g)\in C^\infty(M)\,,$$

or

$$f\otimes g\to\{f,g\}_R=r^{ij}X_ifX_jg\,, \tag{1.1}$$

where $X_if=\varphi(X_i)\rfloor\mathrm{d}f$.

Proposition 1.1. *(1) The operator (1.1) defines a Poisson bracket in $C^\infty(M)$. (2) The brackets $\{\ \}$ and $\{\ \}_R$ form a Poisson pair, so any linear combination $\{\ \}_{a,b}=a\{\ \}+b\{\ \}_R$ is a Poisson bracket.*

The first statement is a direct consequence of the Yang–Baxter equation for R. Let us prove statement (2).

First check that

$$\{\{f,g\},h\}_R+\{\{f,g\}_R,h\}+\circlearrowleft=0\,, \tag{1.2}$$

where $\circlearrowleft$ means summing of all cyclic permutations. Equality (1.2) implies the statement. □

Consider the chain of equalities

$$\begin{aligned}&\{\{f,g\},h\}_R+\{\{f,g\}_R,h\}+\circlearrowleft\\&\quad=r^{ij}(X_i\{f,g\})(X_jh)+r^{ij}\{(X_if),h\}(X_jg)+r^{ij}(X_if)\{(X_jg),h\}+\circlearrowleft\\&\quad=r^{ij}((X_i\{f,g\})(X_jh)+\{(X_if),h\}(X_jg)-(X_jf)\{(X_ig),h\})+\circlearrowleft\,.\end{aligned}$$

Collect similar terms in the last expression, say, those containing X_jh, as a factor. We get

$$r^{ij}((X_i\{f,g\})(X_jh)+\{(X_ig),f\}(X_jh)-(X_jh)\{(X_if),g\}) .$$

The latter expression equals zero because the X_i are Poisson algebra derivations.

If the algebra $\mathfrak{G}$ is generated by Hamiltonian fields, then the bracket $\{\ ,\ \}_R$ and consequently any bracket of the family $\{\ ,\ \}_{a,b}$ allows a restriction on each symplectic leaf of the bracket $\{\ ,\ \}$.

Example 1. A typical example is given by a manifold $M=\mathfrak{G}^*$ with a Poisson–Lie bracket $\{\ ,\ \}$, where $\mathfrak{G}$ is a Lie algebra and R is an R-matrix on this algebra. In this case the family $\{\ ,\ \}_{a,b}$ allows restrictions on any symplectic leaf of the bracket $\{\ ,\ \}$.

Example 2. Another typical example is given by the constant bracket $\{\ ,\ \}$, $\mathfrak{G}$ being a Lie algebra of quadratic Hamiltonians. If a classical R-matrix is given on $\mathfrak{G}$, then the bracket $\{\ ,\ \}_R$ is quadratic. For example,

$$\{f,g\}=\frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial p},\quad \{p,q\}=1,\quad (p;q)\in\mathbb{R}^2,$$
$$f_1=p^2/2,\quad f_2=pq\quad (\{f_1,f_2\}=2f_1),\quad R=f_1\otimes f_2-f_2\otimes f_1 .$$

Then $\{f,g\}_R=p^2\{f,g\}$ (if we take $f_1=p$ then the R-matrix bracket becomes linear: $\{p,q\}=p\{f,g\}$).

Example 3. Let $\mathfrak{G}=\mathrm{sl}(2,\mathbb{R})$ and let X, Y, H be a fixed Chevalley basis in $\mathfrak{G}$ ($[H,X]=2X$, $[H,Y]=-2Y$, $[X,Y]=H$). Suppose x, y, h are the generators in $C^\infty(\mathfrak{G}^*)$ corresponding to X, Y, H. Consider the Poisson–Lie bracket $\{f,g\}(\xi)=\langle[df,dg],\xi\rangle$ on $\mathfrak{G}^*$. The functions $f_1=h$ and $f_2=x^k\varphi(h)$ form a two-dimensional Lie algebra: $\{f_1,f_2\}=2kf_2$. The tensor $R=f_1\otimes f_2-f_2\otimes f_1$ satisfies the classical Yang–Baxter equation. In this case the bracket $\{\ ,\ \}_R$ written in terms of the generators x, y, h has the form

$$\{h,x\}_R=4kx^k\varphi(h),\qquad \{h,y\}_R=-4kx^{k-1}y\varphi(h),$$
$$\{x,y\}_R=2kxh\varphi(h) .$$

Remark. Take $R=X\otimes Y-Y\otimes X$ on the algebra $\mathfrak{G}=\mathrm{sl}(2,\mathbb{R})$. One can easily see that the bracket $\{\ ,\ \}_R$ is a Poisson bracket in $C^\infty(\mathfrak{G}^*)$, although this tensor is not a classical R-matrix. The Poisson–Lie bracket $\{\ ,\ \}$ and the bracket $\{\ ,\ \}_R$ form a Poisson pair.

Suppose now that the initial bracket $\{\ ,\ \}$ is non-degenerate (for example, it is a restriction of the Poisson–Lie bracket on a symplectic leaf). Let Ω be the corresponding symplectic form: $\Omega(X_f, X_g)=\{f, g\}$, where X_f is the Hamiltonian field generated by f.

For a $2n$-dimensional manifold M consider the functional $\xi(f)=\int f\Omega^n$. This functional is central, i.e., it satisfies the condition $\xi(\{f, g\})=0$ for any f, g. Indeed,

$$\int \{f, g\}\Omega^n = \int L_{X_f} g\Omega^n = -\int gL_{X_f}\Omega^n = 0 ,$$

where L_X is the Lie derivative along the field X.

As we have mentioned above, the functional ξ plays an important role in the quantization procedure since it allows one (in some special cases) to consider the space $L_2(\Omega^n)$ as a space of observables.

Recall that, when the Poisson bracket is degenerate, then there are no central functionals. This is exactly the case for the brackets $\{\ ,\ \}_R$.

Consider the pairing,

$$f\otimes g \to \langle f, g\rangle_R = \int \{f, g\}_R \Omega^n$$

on $C^\infty(M)$. Obviously $\langle f, g\rangle_R = -\langle g, f\rangle_R$.

Proposition 1.2. *The pairing $f\otimes g\to\langle f, g\rangle_R$ is a cocycle on the algebra $C^\infty(M)$ with the Poisson bracket $\{\ ,\ \}$, i.e., for any $f, g, h\in C^\infty(M)$ the equality*

$$\langle\{f, g\}, h\rangle_R + \langle\{g, h\}, f\rangle_R + \langle\{h, f\}, g\rangle_R = 0$$

holds.

Proof. Apply the functional ξ to eq. (1.2) and obtain the required equation. □

The following proposition (see ref. [5]) is an algebraic analogue of this statement.

Proposition 1.3. *Let $\mathfrak{G}$ be a Lie algebra with an invariant pairing*

$$\langle\ ,\ \rangle : \mathfrak{G}^{\otimes 2} \to k \ (X\otimes Y \to \langle X, Y\rangle) .$$

Then the pairing

$$\langle\ ,\ \rangle_R = \langle\ ,\ \rangle R \ (\langle X, Y\rangle_R = \langle R(X\otimes Y)\rangle) ,$$

where $R(X\otimes Y) = r^{ij}[X_i, X]\otimes[X_j, Y]$ and $r^{ij}X_i\otimes X_j$ is an element of $\Lambda^2\mathfrak{G}$, is a cocycle on the Lie algebra $\mathfrak{G}$, i.e., the relation

$$\langle [X, Y], Z\rangle_R + \langle [Y, Z], X\rangle_R + \langle [Z, X], Y\rangle_R = 0$$

holds.

The proof is similar to that of proposition 1.1. First the equality

$$r^{ij}(\langle [X_i, [X, Y]], [X_j, Z]\rangle + \langle [[X_i, X], Z], [X_j, Y]\rangle - \langle [X_j, X], [[X_i, Y], Z]\rangle) + \circlearrowleft = 0$$

is verified. The sum of the second and the third terms and their permutations equals zero since the pairing $\langle\ ,\ \rangle$ is invariant. □

Remark. Note that this statement remains valid if the element $r^{ij}X_i \otimes X_j \in \Lambda^2 \mathfrak{G}$ is replaced by an element of $\Lambda^2 D(\mathfrak{G})$ where $D(\mathfrak{G})$ is the derivation algebra of $\mathfrak{G}$.

It should be emphasized that not all cocycles on the Lie algebra $\mathfrak{G}$ can be represented in the form $\langle\ ,\ \rangle_R$ with a tensor $R \in \Lambda^2 D(\mathfrak{G})$.

2. Quantization of Poisson brackets associated with *R*-matrices

Recall that the first step of a "Poisson bracket quantization" is a deformation quantization, i.e., a construction of a family of associative structures in $C^\infty(M)$ depending on the Planck parameter $\hbar$ and satisfying the correspondence principle

$$f *_\hbar g|_{\hbar=0} = f \cdot g, \qquad \lim_{\hbar \to 0} (f *_\hbar g - g *_\hbar f)\hbar^{-1} = \{f, g\} .$$

We denote by $*_\hbar$ multiplication corresponding to the parameter $\hbar$. Strictly speaking, the element $f *_\hbar g$ belongs to the space $C^\infty(M)[[\hbar]]$ of formal power series of $\hbar$, but we omit this dependence on $\hbar$ in our notations. We suppose also that the functions $f = 1$ is a two-sided unit in the deformation algebras.

In this section we study the problem of simultaneous quantization of the family $\{\ ,\ \}_{a,b} = a\{\ ,\ \} + b\{\ ,\ \}_R$, where $\{\ ,\ \}$ is a given bracket on the manifold M and $\{\ ,\ \}_R$ is the R-matrix bracket constructed in the previous section.

Suppose that the deformation quantization for the initial bracket $\{\ ,\ \}$ has already been constructed. Denote the multiplication in the corresponding associative algebra by $\circ_\hbar$ and the space $C^\infty(M)$ equipped with this multiplication by $C^\infty(M, \circ_\hbar)$.

Suppose also that the action of the Lie algebra $\mathfrak{G}$ extends to an action on this algebra. Thus we get a representation $\varphi : \mathfrak{G} \to \mathrm{Der}\, C^\infty(M, \circ_\hbar)$ of $\mathfrak{G}$ in the derivation algebra of the algebra $C^\infty(M, \circ_\hbar)$.

Consider the series

$$F_\hbar(X, X) = 1 \otimes 1 + \tfrac{1}{2}\hbar r^{ij} X_i \otimes X_j + \sum_{|\alpha| + |\beta| \geq 2} P_{\alpha,\beta}(\hbar) X^\alpha \otimes X^\beta ,$$

quantizing the given R-matrix in the sense of ref. [6] (it is clear that the quantizing series of the initial R-matrix is not unique). Here $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and $X^\beta = X_1^{\beta_1} \cdots X_n^{\beta_n}$ are elements of the enveloping algebra $U(\mathfrak{G})$ of $\mathfrak{G}$.

This series satisfies the following conditions:

$$F_\hbar(X,0) = F_\hbar(0,X) = 1 ,$$
$$F_\hbar(X+Y,Z) F_\hbar(X,Y) = F_\hbar(X,Y+Z) F_\hbar(Y,Z) .$$

Consider the map $F_\hbar : \mathrm{C}^\infty(M)^{\otimes 2} \to \mathrm{C}^\infty(M)^{\otimes 2}$ given by

$$F_\hbar(f \otimes g) = f \otimes g + \tfrac{1}{2} \hbar r^{ij} X_i f \otimes X_j g + \sum_{\alpha,\beta} P_{\alpha,\beta}(\hbar) X^\alpha f \otimes X^\beta g . \tag{2.1}$$

Here again we write Xf instead of $\varphi(X) \rfloor \mathrm{d}f$.

Let $f *_{\hbar,\nu} g = \circ_\hbar F_\nu(f \otimes g)$. This multiplication can be rewritten as follows:

$$f *_{\hbar,\nu} g = f \circ_\hbar g + \tfrac{1}{2} \nu r^{ij} X_i f \circ_\hbar X_j g + \sum_{\alpha,\beta} P_{\alpha,\beta}(\nu) X^\alpha f \circ_\hbar X^\beta g .$$

Proposition 2.1. *The multiplication $*_{\hbar,\nu}$ is associative and the function $f \equiv 1$ is the unit in $\mathrm{C}^\infty(M, *_{\hbar,\nu})$.*

This proposition will be discussed in section 4.

We shall write $*_{a,b}$ instead of $*_{a\hbar,b\hbar}$. In this case we suppose that a, b, $\hbar$, $\nu \in \mathbb{R}$ (we write $\hbar\mathrm{i}$ and $a\mathrm{i}$ if we want the first parameter to be purely imaginary).

The following proposition is obvious.

Proposition 2.2. *The correspondence principle holds, i.e.,*

$$f *_{a,b} g|_{\hbar=0} = fg, \qquad \lim_{\hbar \to 0} (f *_{a,b} g - g *_{a,b} f) \hbar^{-1} = \{f, g\}_{a,b} ,$$

so $(\mathrm{C}^\infty(M, *_{a,b})$ is a deformation algebra for the Poisson algebra $\mathrm{C}^\infty(M, \{\ ,\ \}_{a,b})$.

Remark. Let $\nu = \hbar^2$ in the definition of the associative multiplication $*_{\hbar,\nu}$. Then

$$\lim_{\hbar \to 0} (f *_{\hbar,\hbar^2} g - g *_{\hbar,\hbar^2} f) \hbar^{-1} = \{f, g\} .$$

This gives a new quantization for the initial Poisson bracket.

Now consider a representation

$$\tau : \mathrm{C}^\infty(M, \circ_{\mathrm{i}\hbar}) \to \mathrm{End}(V) \quad [\tau(f) = A_f = A_f(\hbar)]$$

of the associative algebra $\mathrm{C}^\infty(M, \circ_{\mathrm{i}\hbar})$ in a complex Hilbert space V equipped with a pairing $\langle\ ,\ \rangle : V^{\otimes 2} \to \mathrm{C}$ such that $A_{\bar f} = A_f^*$.

Consider a representation $\varphi : \mathfrak{G} \to \mathrm{End}(V)$ of $\mathfrak{G}$ in the same space such that

$$\langle \varphi(X)v_1, v_2\rangle + \langle v_1, \varphi(X)v_2\rangle = 0 ,$$

$$\varphi(X)(A_f v) = A_{X(f)}v + A_f\varphi(X)v, \quad X\in\mathfrak{G} .$$

Construct a map $f\to A_f^{a,b}\in \mathrm{End}(V)$,

$$A_f^{a,b}v = \mathrm{ev}\, E_{b\hbar}(A_f(a\hbar)\otimes v) .$$

Here $F_\hbar$ is an endomorphism of the tensor product of two left $\mathfrak{G}$-modules defined by a formula similar to (2.1) and ev : $A\otimes v\to Av$ is the evaluation map for the operator A and the element v. In section 4 we shall prove the mapping $f\to A_f^{a,b}$ to be a representation of $C^\infty(M, *_{a,b})$. The properties of this representation will be expressed in terms of the symmetry $S = F_{b\hbar}^{-1}\cdot\sigma\cdot F_{b\hbar}$ and the deformed pairing $\langle\ ,\ \rangle_S = \langle\ ,\ \rangle F_{b\hbar}$.

In particular, consider the adjoint representation $f\to A_f$ of $C^\infty(M, \circ_{i\hbar})$, where $A_f g = f\circ_{i\hbar} g$. Suppose that the operators A_f are self-adjoint for real valued $f\in C^\infty(M)$ (with respect to the pairing $f\otimes g\to\int f\bar{g}\Omega^n$). In this case the pairing deformation has the form

$$\langle f, g\rangle_S = \int F_{b\hbar}(f\otimes\bar{g})\Omega^n .$$

The linear term of this deformation is $\int R(f\otimes\bar{g})\Omega^n$ (up to a constant) and on the space of real functions it coincides with the R-matrix cocycle described in section 1.

Example. It is known that the algebra $C^\infty(\mathbb{R}, \circ_{i\hbar})$ with the multiplication

$$f\circ_{i\hbar} g = \mu \exp\left[\tfrac{1}{2}i\hbar\left(\frac{\partial}{\partial p}\otimes\frac{\partial}{\partial q} - \frac{\partial}{\partial q}\otimes\frac{\partial}{\partial p}\right)\right](f\otimes g)$$

is a result of the Weyl quantization of the bracket

$$\{f, g\} = \frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial p}, \quad (p, q)\in\mathbb{R}^2 .$$

The reader can easily verify that the operators of the adjoint representation satisfy the condition $A_{\bar{f}} = A_f^*$ for the Hilbert pairing $f\otimes g\to\int f\bar{g}\Omega^n$, where $\Omega = \mathrm{d}p\wedge\mathrm{d}q$.

Let $\mathfrak{G}$ and R be the same as in example 2 of section 1 and either $f_1 = p^2/2, f_2 = pq$ (case 1) or $f_1 = p, f_2 = pq$ (case 2). One can easily see that the Hamiltonian fields generated by f_1 and f_2 (in both cases) are derivations of $C^\infty(\mathbb{R}^2, \circ_{i\hbar})$ and thus we can construct, using the above-mentioned method, the deformation quantization for the family $ai\{\ ,\ \} + b\{\ ,\ \}_R$, where $\{\ ,\ \}_R = p^2\{\ ,\ \}$ in case 1 and $\{\ ,\ \}_R = p\{\ ,\ \}$ in case 2.

Remark. It is possible to consider multi-parameter families of brackets. For example, if there are three Poisson brackets $\{\ ,\ \}$, $\{\ ,\ \}'$, $\{\ ,\ \}''$ forming a Poisson triple and any bracket of the family $b\{\ ,\ \}'+c\{\ ,\ \}''$ is an R-matrix one, then under similar assumptions on the quantization of the first bracket, it is possible to introduce a simultaneous quantization for the family $a\{\ ,\ \}+b\{\ ,\ \}'+c\{\ ,\ \}''$ in $C^\infty(M)$. The brackets

$$\{f,g\}=\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}, \qquad \{\ ,\ \}'=p\{\ ,\ \}, \quad \{\ ,\ \}''=q\{\ ,\ \},$$

serve as an example of such a triple.

3. "S-quantum mechanics": a general scheme

The quantum mechanics scheme is based on the hypothesis that observables are self-adjoint operators in a complex Hilbert space. The set of all such operators is a Lie algebra with the operation $A\otimes B\to \mathrm{i}(A\circ B-B\circ A)$. A trace is defined on a dense set of such operators.

A state is a self-adjoint nuclear positive operator $M: V\to V$ such that $\operatorname{tr} M=1$. The value of an observable A in a state M is measured by $\operatorname{tr} A\circ M$. A fixed observable H, which is called a Hamiltonian (or energy), defines the dynamics of other observables by the equation (Heisenberg picture)

$$\mathrm{d}A(t)/\mathrm{d}t=(\mathrm{i}/\hbar)\,[H,A(t)]\,. \tag{3.1}$$

We extend some features of this scheme to the "S-quantum mechanics". (We omit some difficult problems, concentrating on the notation of positivity.) Consider a linear space V over the field $\mathbb{R}$. Let a symmetry $S: V^{\otimes 2}\to V^{\otimes 2}$ be defined in $V^{\otimes 2}$ and let $\langle\ ,\ \rangle: V^{\otimes 2}\to\mathbb{R}$ be a non-degenerate pairing, satisfying two axioms:

(1) $\langle\ ,\ \rangle=\langle\ ,\ \rangle S$ (S-symmetry);

(2) $\langle\ ,\ \rangle^{(1)}=\langle\ ,\ \rangle^{(2)}S^{(1)}S^{(2)}$ (S-invariance).

Here and below we write

$$S^{(1)}=S\otimes\mathrm{id},\quad S^{(2)}=\mathrm{id}\otimes S,$$
$$\langle\ ,\ \rangle^{(1)}=\langle\ ,\ \rangle\otimes\mathrm{id},\quad \langle\ ,\ \rangle^{(2)}=\mathrm{id}\otimes\langle\ ,\ \rangle\,,$$

and so on. To avoid topological problems we suppose $\dim V<\infty$ and identify V and V^* with the help of the pairing $\langle\ ,\ \rangle$.

Fix a basis $\{e_i\}$ in V. Write $\langle e_i, e_j\rangle=g_{ij}\in\mathbb{R}$. Denote by $V_{\mathbb{C}}$ the complexification of V. Extend the symmetry and the pairing to $V_{\mathbb{C}}^{\otimes 2}$ in the usual manner, for example, $\langle ae_p, be_q\rangle=a\bar{b}g_{pq}$, if $a, b\in\mathbb{C}$.

Below we use the notation: $V_{\mathbb{C}}=V$; the initial real space is denoted by $V_{\mathbb{R}}$. All objects (operators, tensors) will be considered over $\mathbb{C}$.

A non-degenerate pairing in the space V allows us to identify $V^{\otimes 2}$ with End(V). Introduce a homomorphism $A_N: V \to V$ ($N = a^{ij} e_i \otimes e_j \in V^{\otimes 2}$):

$$A_N e_p = a^{ij} e_i \langle e_j, e_p \rangle = a^{ij} g_{jp} e_i \, .$$

The correspondence $N \to A_N$ is bijective. So we can introduce an associative algebraic structure in $V^{\otimes 2}$.

Define the S-trace and the S-conjugation operator on $V^{\otimes 2}$:

$$\mathrm{tr}_S N = \langle \, , \, \rangle_N, \quad \mathrm{Adj}_S N = \overline{S(N)} \, .$$

Let

$$\mathrm{tr}_S A_N = \mathrm{tr}_S N, \quad A_N^* = \mathrm{Adj}_S A_N = A_{\mathrm{Adj}_S N} \, .$$

Obviously $\mathrm{Adj}_S^2 = \mathrm{id}$.

Proposition 3.1. *The following equalities hold:*
(1) $\mathrm{tr}_S A_N^* = \mathrm{tr}_S A_N$;
(2) $\langle A_N e_p, e_q \rangle = \langle A_N^* \tilde{e}_q, \tilde{e}_p \rangle$, where $\tilde{e}_q \otimes \tilde{e}_p = S(e_p \otimes e_q)$.

The first statement is obvious. The second one is implied by the following lemma.

Lemma 3.2. *The equality*

$$\langle \, , \, \rangle \langle \, , \, \rangle^{(2)} S^{(1)} = \langle \, , \, \rangle \langle \, , \, \rangle^{(2)} S^{(3)} \qquad (3.2)$$

holds on $V_{\mathbb{R}}^{\otimes 4}$.

Proof. This relation is implied by the equality

$$\langle \, , \, \rangle^{(2)} S^{(1)} = \langle \, , \, \rangle^{(1)} S^{(2)} \, ,$$

which is a reformulation of the S-invariance of the pairing. □

The second statement of proposition 3.1 is equivalent to the adjoint operator definition.

Denote by $SA(S)$ the set of all self-adjoint operators, i.e., the operators satisfying the condition $A = A^* = \mathrm{Adj}_S A$.

Consider the matrix g^{kl} inverse to the matrix g_{ij}: $g^{kl} g_{li} = \delta_i^k$, and the "embedding of the unit" operator $1 \to g = g^{ij} e_i \otimes e_j$, which is adjoint to the pairing operator. The pairing $V^{\otimes 2} \otimes V^{\otimes 2} \to \mathbb{C}$ is defined by the formula $\langle e_i \otimes e_j, e_k \otimes e_l \rangle = g_{il} g_{jk}$.

Proposition 3.3. *The embedding of the unit operator is S-invariant, i.e., the element* $g = g^{ij} e_i \otimes e_j$ *is central:* $S(e_p \otimes g) = g \otimes e_p$.

Note that $A_g = \mathrm{id} \in SA(S)$.

Define a symmetry $S: \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2}$ by identifying $\mathrm{End}(V)^{\otimes 2}$ with $V^{\otimes 4}$ [we transpose $(a \otimes b)$ and $(c \otimes d)$ in $(a \otimes b) \otimes (c \otimes d)$ with the help of the operator $S^{(2)} S^{(1)} S^{(3)} S^{(2)}$].

Remark. In the theory of monoidal quasi-tensor categories the trace in the operator algebra is defined independently of any pairing (see ref. [3]). One can show that for an S-invariant, S-symmetric pairing both definitions are equivalent. Therefore the notion of the S-trace does not depend upon the pairing. On the contrary, the operator Adj depends upon the pairing.

There exists an operator B such that $\mathrm{tr}_S A = \mathrm{tr}\, AB$. This operator can be found from the equality $\langle e_p, B e_q \rangle = g_{qp}$.

Note that the quantity $\mathrm{tr}_S\, \mathrm{id}$ is integer and equals p if S is an even symmetry of rank p in the sense of section 5.

Proposition 3.4. *The functional* tr_S *is S-central, i.e.* $\mathrm{tr}_S A \circ B = \mathrm{tr}_S S(A \otimes B)$, A, $B \in \mathrm{End}(V)$.

We omit a proof of this simple statement.

Proposition 3.5. *The conjugation operator is S-invariant, i.e.*, $S(\mathrm{Adj}_S \otimes \mathrm{id}) = (\mathrm{id} \otimes \mathrm{Adj}_S) S$.

This statement is a direct consequence of QYBE.

Proposition 3.6. *The quality*

$$\mathrm{Adj}_S(A \circ B) = \circ\, (\mathrm{Adj}_S \otimes \mathrm{Adj}_S) S(A \otimes B)$$

holds.

Actually, we need to prove that

$$S\langle\ ,\ \rangle^{(2)} = \langle\ ,\ \rangle^{(2)} (S^{(3)} S^{(1)}) (S^{(2)} S^{(3)} S^{(1)} S^{(2)}) . \tag{3.3}$$

Because of the relation $S\langle\ ,\ \rangle^{(2)} = \langle\ ,\ \rangle^{(1)} S^{(3)} S^{(2)} S^{(1)}$ (which a is consequence of the second property in the definition of the pairing $\langle\ ,\ \rangle$), (3.3) is equivalent to the following equality:

$$\langle\ ,\ \rangle^{(1)} = \langle\ ,\ \rangle^{(2)} (S^{(3)} S^{(1)}) (S^{(2)} S^{(3)} S^{(1)} S^{(2)}) (S^{(1)} S^{(2)} S^{(3)}) .$$

Now, the required equality is implied by the following chain of relations:

$$\begin{aligned}
&\langle\,,\,\rangle^{(2)}(S^{(3)}S^{(1)})(S^{(2)}S^{(3)}S^{(1)}S^{(2)})(S^{(1)}S^{(2)}S^{(3)}\\
&=\langle\,,\,\rangle^{(2)}S^{(3)}S^{(1)}S^{(2)}S^{(3)}(S^{(1)}S^{(2)}S^{(1)}S^{(2)})S^{(3)}\\
&=\langle\,,\,\rangle^{(2)}S^{(3)}S^{(1)}S^{(2)}S^{(3)}S^{(2)}S^{(1)}S^{(3)}\\
&=\langle\,,\,\rangle^{(2)}S^{(3)}S^{(1)}(S^{(2)}S^{(3)}S^{(2)}S^{(3)})S^{(1)}\\
&=\langle\,,\,\rangle^{(2)}S^{(3)}S^{(1)}S^{(3)}S^{(2)}S^{(1)}=\langle\,,\,\rangle^{(2)}S^{(1)}S^{(2)}S^{(1)}\\
&=\langle\,,\,\rangle^{(2)}S^{(2)}S^{(1)}S^{(2)}=\langle\,,\,\rangle^{(2)}S^{(1)}S^{(2)}=\langle\,,\,\rangle^{(1)}\,.
\end{aligned}$$
□

Proposition 3.7. *If* A, $B\in SA(S)$, *then* $A\circ B+\circ S(A\otimes B)\in SA(S)$, *and* $\mathrm{i}(A\circ B-\circ S(A\otimes B))\in SA(S)$.

Proof is immediately provided by propositions 3.5 and 3.6. What kinds of operators $A\in SA(S)$ should be regarded as Hamiltonians in our theory? In supermathematics this role is played by even self-adjoint operators (due to the evenness of the time variable in the Schrödinger equation). Denote by $Z(S)$ the set of all S-self-adjoint S-variant operators, i.e., operators $A\in SA(S)$ satisfying the condition $S(A\otimes e_p)=e_p\otimes A$. We consider these operators to be the analogues of even operators in "S-mechanics".

Consider the dynamic equation (3.1) with the Hamiltonian $H\in Z(S)$. The commutator $[H,A(t)]$ can be changed to the S-commutator, i.e., to $[H,A(t)]_S=H\circ A-\circ S(H\otimes A)$, since the relation $S(H\otimes A)=A\otimes H$ holds for $H\in Z(S)$.

The dynamics of the observable $A(t)$ is expressed by the classical formula

$$A(t)=\mathrm{e}^{\mathrm{i}Ht/\hbar}A(0)\mathrm{e}^{-\mathrm{i}Ht/\hbar}\,.$$

Let $\mathrm{Spec}(H)$ be the spectrum of the operator $H\in Z(S)$ and $\lambda\in\mathrm{Spec}(H)$. Denote by V_λ the eigenspace of V corresponding to the eigenvalue λ. We assume that V is a direct sum,

$$V=\bigoplus_{\lambda\in\mathrm{Spec}(H)}V_\lambda\,.$$

Proposition 3.8. $S(V_\lambda\otimes V_\mu)=V_\mu\otimes V_\lambda$.

Proof. Let $x\in V_\lambda$, $y\in V_\mu$. Apply the operator $\mathrm{id}\otimes H$ to the element $S(x\otimes y)$. Taking into consideration that H is S-invariant we get

$$(\mathrm{id}\otimes H)S(x\otimes y)=S(Hx\otimes y)=\lambda S(x\otimes y)\,.$$

Since $V=\oplus_{\lambda\in\mathrm{Spec}(H)}V_\lambda$, we conclude that $S(x\otimes y)=V\otimes V_\lambda$. Applying the operator $H\otimes\mathrm{id}$ to the element $S(x\otimes y)$ we get the required statement. □

Thus, we have expanded $V^{\otimes 2}$ into a direct sum of spaces,

$$V^{\otimes 2} = \bigoplus_{\lambda,\mu \in \mathrm{Spec}(H)} (V_\lambda \otimes V_\mu) .$$

Corrolary 3.9. *If $A = A_N \in SA(S)$, then $N \in \oplus_\lambda (V_\lambda \otimes V_\lambda)$.*

Proofs of the following statements do not differ from their classical counterparts.

Proposition 3.10. *The subspaces V_λ and V_μ $(\lambda \neq \mu)$ are orthogonal with respect to the pairing $\langle\ ,\rangle$ and, consequently, the pairing $\langle\ ,\ \rangle : V_\lambda^{\otimes 2} \to \mathbb{C}$ is non-degenerate. Moreover,* $\mathrm{Spec}(H) \subset \mathbb{R}$.

Call $M \in SA(S)$ an H-state if $M = \sum_{\lambda \in \mathrm{Spec}(H)} a_\lambda M_\lambda$, where $M_\lambda \in V_\lambda^{\otimes 2}$, $\mathrm{tr}_S M_\lambda = 1$, $a_\lambda \in \mathbb{R}$, $a_\lambda > 0$, $\sum_\lambda a_\lambda = 1$. Obviously, H-states form a convex cone.

The value of the observable A in the state M is measured by $\mathrm{tr}_S A \circ M$.

Calculate the value of the Hamiltonian H in the state M_λ. Since the Hamiltonian H is scalar on each V_λ, H corresponds to the tensor $\lambda g_\lambda^{ij} e_i \otimes e_j$, where $g_\lambda^{ij} e_i \otimes e_j$ is the tensor that is "inverse" to the bilinear form $\langle\ ,\ \rangle : V_\lambda^{\otimes 2} \to \mathbb{C}$. Hence $\mathrm{tr}_S H \circ M_\lambda = \lambda\, \mathrm{tr}_S M_\lambda = \lambda$.

If the "vacuum" state $M = \mathrm{Vac}$ is defined for a given quantum system, then we can study "correlation" functions of the type:

$$\langle A_1(t_1) \cdots A_n(t_n) M \rangle = \mathrm{tr}_S A_1(t_1) \cdots A_n(t_n) M .$$

4. "S-quantum mechanics": quasi-classical case

In section 3 we have presented the general scheme for the "S-quantum mechanics". Here we demonstrate that the representation $f \to A_f \in \mathrm{End}(V)$ constructed in section 2 is a part of this scheme.

As was mentioned above, one can construct a map $F_\nu : V_1 \otimes V_2 \to V_1 \otimes V_2$ for any two left $\mathfrak{G}$-modules V_1 and V_2 similarly to (2.1).

Let $V = V_1 = V_2$ be complex Hilbert spaces equipped with a pairing $\langle\ ,\ \rangle : V^{\otimes 2} \to \mathbb{C}$. Fix $\nu = b\hbar$ and put $S = F_\nu^{-1} \circ \sigma \circ F_\nu$, where F_ν^{-1} is a series in ν which is formally inverse to F_ν and $\langle\ ,\ \rangle_S = \langle\ ,\ \rangle F_\nu$.

Proposition 4.1. *The following statements hold:*

(1) *S is a YB symmetry;*

(2) *the pairing $\langle\ ,\ \rangle_S$ is S-invariant and S-symmetric;*

(3) *the map $f \to A_f^{a,b}$ is a representation of the algebra* $C^\infty(M, *_{a,b})$;

(4) *$A_{\bar f}^{a,b} = \mathrm{Adj}_S A_f^{a,b}$, where $\mathrm{Adj}_S A = A^*$ for all $A \in \mathrm{End}(V)$ is defined by the*

equality

$$\langle A^* e_i, e_j \rangle = \overline{\langle A\tilde{e}_j, \tilde{e}_i \rangle}, \quad e_i \in V, \quad \tilde{e}_j \otimes \tilde{e}_i = S_\nu(e_i \otimes e_j) .$$

Prove that S satisfies the QYBE (the unitarity of S is obvious). Let σ be the usual permutation in $V^{\otimes 2}$. Since

$$F_\nu(\sigma F_\nu(x \otimes y) \otimes z) = \sigma^{(1)} F_\nu(F_\nu(x \otimes y) \otimes z) ,$$

the following relation holds:

$$S^{(1)} = F_\nu^{(1)-1} \circ \sigma^{(1)} \circ F_\nu^{(1)} = {}_3F_\nu^{-1} \circ \sigma^{(1)} \circ {}_3F_\nu ,$$

where

$${}_3F_\nu(x \otimes y \otimes z) = F_\nu(x \otimes F_\nu(y \otimes z)) = F_\nu(F_\nu(x \otimes y) \otimes z)) .$$

In this way $S^{(1)}$ and $\sigma^{(1)}$ are intertwined by ${}_3F_\nu$. The operators $S^{(1)}$ and $\sigma^{(2)}$ are intertwined in the same manner. This provides the QYBE for the operator S.

Proposition 2.1 and statements 2 and 4 of this proposition can be proved in the same way. Prove now statement 3.

Denote by φ_F the map $f \to A_f^{a,b}$, i.e.,

$$\varphi_F(f)v = \mathrm{ev}(\varphi \otimes \mathrm{id}) F_\nu(f \otimes v) .$$

Here $\varphi : f \to A_f$ is a representation of the algebra $C^\infty(M, \circ_{i\hbar})$. Then the following chain of equalities implies statement 3:

$$\begin{aligned}
\varphi_F(f *_{a,b} g) &= \mathrm{ev}(\varphi \otimes \mathrm{id}) F_\nu((f *_{a,b} g) \otimes v) \\
&= \mathrm{ev}(\varphi \otimes \mathrm{id}) F_\nu(\circ_{ai\hbar} F_\nu(f \otimes g) \otimes v) \\
&= \mathrm{ev}(\varphi \otimes \mathrm{id})(\circ_{ai\hbar} \otimes \mathrm{id}) F_\nu(F_\nu(f \otimes g) \otimes v) \\
&= \mathrm{ev}(\mathrm{mult} \otimes \mathrm{id})(\varphi \otimes \varphi \otimes \mathrm{id}) F_\nu((F_\nu(f \otimes g) \otimes v) \\
&= \mathrm{ev}(\varphi \otimes \mathrm{id}) F_\nu(f \otimes \varphi_F(g) v) = \varphi_F(f)(\varphi_F(g) v) .
\end{aligned}$$

Here "mult" denotes the multiplication operator $A \otimes B \to A \circ B$.

Thus the algebra quantizing the brackets $\{\ ,\ \}$ is represented by operators $A_f^{a,b} : V \to V$, where V is equipped with symmetry S and the S-symmetric pairing. The general scheme of such quantum objects was presented in section 3. It is the existence of the operators F_ν intertwining the standard quantum mechanics and "S-quantum mechanics" that provides specific features characterizing the objects of the present section.

S-self-adjoint operators do not form a Lie algebra but they do form an S-Lie algebra. This notion was introduced by the first author in ref. [7]. Let us recall it.

Definition. A linear space $\mathfrak{G}$ is called an S-Lie algebra if $\mathfrak{G}$ is equipped with a symmetry $S: \mathfrak{G}^{\otimes 2} \to \mathfrak{G}^{\otimes 2}$ and a "S-Lie bracket" $[\ ,\ ]: \mathfrak{G}^{\otimes 2} \to \mathfrak{G}$, i.e., a map satisfying the conditions

(1) $[\ ,\ ]S = -[\ ,\ ]$ (S-skew-symmetry);

(2) $[\ ,\ ][\ ,\ ]^{(1)}(\mathrm{id} + S^{(1)}S^{(2)} + S^{(2)}S^{(1)}) = 0$ (S-Jacobi identity);

(3) $S[\ ,\ ]^{(1)} = [\ ,\ ]^{(2)}S^{(1)}S^{(2)}$ (S-invariance).

A pairing $\langle\ ,\ \rangle: \mathfrak{G}^{\otimes 2} \to k$ is called invariant if it satisfies the following conditions:

(1) $\langle\ ,\ \rangle S = \langle\ ,\ \rangle$ (S-symmetry);

(2) $\langle\ ,\ \rangle^{(1)} = \langle\ ,\ \rangle^{(2)}S^{(1)}S^{(2)}$ (S-invariance);

(3) $\langle\ ,\ \rangle[\ ,\ ]^{(2)} + \langle\ ,\ \rangle[\ ,\ ]^{(2)}S^{(1)} = 0$ ($\mathfrak{G}$-invariance).

Consider the S-Lie algebra generated by S-self-adjoint operators $A_f^{a,b}$. This S-Lie algebra is equivalent to the Lie algebra of self-adjoint operators with the standard commutator. Recall (see ref. [7]) that two S-Lie algebras are equivalent if they can be intertwined by an operator $F = ({}_pF)$ (the first two diagrams below are commutative).

In this paper the operator F intertwines also the pairings defined on these algebras and therefore these algebras are U-equivalent in the following sense.

Two S-Lie algebras equipped with invariant pairings $\mathfrak{G}_1 = (S_1, [\ ,\ ]_1, \langle\ ,\ \rangle_1)$ and $\mathfrak{G}_2 = (S_2, [\ ,\ ]_2, \langle\ ,\ \rangle_2)$ are unitarily (or U-) equivalent if there exist operators ${}_pF: \mathfrak{G}_1^{\otimes p} \to \mathfrak{G}_2^{\otimes p}$, $p \geq 0$, such that the following diagrams are commutative:

$$\begin{tikzcd} \mathfrak{G}_1^{\otimes p} \arrow[r, "S_1^{(i)}"] \arrow[d, "{}_pF"'] & \mathfrak{G}_1^{\otimes p} \arrow[d, "{}_pF"] \\ \mathfrak{G}_2^{\otimes p} \arrow[r, "S_2^{(i)}"] & \mathfrak{G}_2^{\otimes p} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{G}_1^{\otimes p} \arrow[r, "{[\ ,\ ]_1^{(i)}}"] \arrow[d, "{}_pF"'] & \mathfrak{G}_1^{\otimes p-1} \arrow[d, "{}_{p-1}F"] \\ \mathfrak{G}_2^{\otimes p} \arrow[r, "{[\ ,\ ]_2^{(i)}}"] & \mathfrak{G}_2^{\otimes p-1} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{G}_1^{\otimes p} \arrow[r, "{\langle\ ,\ \rangle_1^{(i)}}"] \arrow[d, "{}_pF"'] & \mathfrak{G}_1^{\otimes p-2} \arrow[d, "{}_{p-2}F"] \\ \mathfrak{G}_2^{\otimes p} \arrow[r, "{\langle\ ,\ \rangle_2^{(i)}}"] & \mathfrak{G}_2^{\otimes p-2} \end{tikzcd}$$

(we suppose that ${}_0F = \mathrm{id}$, ${}_1F = \mathrm{id}$, i.e., $\mathfrak{G}_1$ coincides with $\mathfrak{G}_2$ as a linear space).

The pairing $\langle\ ,\ \rangle_{a,b}$ on the S-algebra generated by S-self-adjoint operators $A_f^{a,b}$ is determined by the S-trace,

$$\langle A_f^{a,b}, A_g^{a,b} \rangle = \mathrm{tr}_S A_f^{a,b} \circ A_g^{a,b}\,.$$

Consider a dynamic model with a Hamiltonian f. Let G be a Lie symmetry group for this model, $\mathfrak{G}$ be its Lie algebra and $R \in \Lambda^2\mathfrak{G}$ be an R-matrix. Since f is G-invariant and therefore $Xf = 0$ for all $X \in \mathfrak{G}$ we obtain $A_f^{a,b} = A_f^{a,0} = A_f(a\hbar)$. So $\mathrm{Spec}(A_f(a\hbar)) = \mathrm{Spec}(A_f^{a,b})$, i.e., the intertwining does not change the spectrum of a dynamic model if the 'intertwining group" is the symmetry group of the Hamiltonian. In the next section we show that the statistical sum of an "S-quantum oscillator" depends on a Poincaré series of a symmetry S only (an intertwining does not change this series).

Now we shall quantize the cocycle described in proposition 1.3.

Proposition 4.2. *Let F be the same series as above, quantizing a classical R-matrix R of proposition 1.3. Let $F_\hbar = \mathrm{id} + \frac{1}{2}\hbar R + \cdots$ be above-mentioned series and $F_\hbar : \mathfrak{G}^{\otimes 2} \to \mathfrak{G}^{\otimes 2}$ be the corresponding operator:*

$$F_\hbar(X \otimes Y) = X \otimes Y + \tfrac{1}{2}\hbar R(X \otimes Y) + \cdots .$$

Consider the operators: $S_\hbar = F_\hbar^{-1} \circ \sigma \circ F_\hbar$, $[\ ,\ ]_S = [\ ,\ ]F_\hbar$ and $\langle\ ,\ \rangle_S = \langle\ ,\ \rangle F_\hbar$. Then $\mathfrak{G}$ is an S-Lie algebra with respect to S and $[\ ,\ ]_S$ and the pairing $\langle\ ,\ \rangle_S : \mathfrak{G}^{\otimes 2} \to k$ is invariant.

The proof is similar to the one above and is left to the reader.

So an invariant pairing on an S-Lie algebra is the quantum analogue of an R-matrix cocycle on an initial Lie algebra $\mathfrak{G}$.

5. Example: "S-quantum oscillator"

In this section we construct examples of S-quantum systems without assuming S to be quasi-classical.

Consider a space V over $\mathbb{R}$ equipped with a symmetry $S : V^{\otimes 2} \to V^{\otimes 2}$ and a pairing $\langle\ ,\ \rangle : V^{\otimes 2} \to \mathbb{R}$. Let S and $\langle\ ,\ \rangle$ satisfying the conditions of section 3. Fix a basis $\{e_i\}$ and write $\langle e_i, e_j \rangle = g_{ij}$. Introduce "$S$-symmetric" and "$S$-exterior" algebras

$$\Lambda_+(V) = T/I_-, \qquad \Lambda_-(V) = T/I_+ ,$$

as quotient spaces of the free tensor algebra $T = T(V)$ over the ideals $I_- = \{e_i \otimes e_j - S(e_i \otimes e_j)\}$ and $I_+ = \{e_i \otimes e_j + S(e_i \otimes e_j)\}$.

Consider the Poincaré series $\mathscr{P}_\pm(t) = \mathscr{P}_\pm(t, V) = \sum_{i \geq 0} r_\pm^{(i)} t^i$ for $\Lambda_\pm(V)$, where $r_\pm^{(i)} = \dim \Lambda_\pm^i(V)$ and $\Lambda_\pm^i(V)$ is the homogeneous part of degree i of $\Lambda_\pm(V)$. For $S = \sigma$ (σ is an ordinary permutation) $\mathscr{P}_-(t)$ is a polynomial of degree n and $r_-^{(i)} = \binom{n}{i}$ (the "classical" case).

A symmetry S is called even of rank p if $\mathscr{P}_-(t)$ is a polynomial of degree p and $r_-^{(p)} = 1$.

In ref. [4] it was proved that for any linear space V ($\dim V = n \geq 2$) and integer p such that $2 \leq p \leq n$ there exists an even symmetry $S : V^{\otimes 2} \to V^{\otimes 2}$ of rank p.

Obviously if $p \neq n$, such a symmetry cannot be obtained by deformation of an ordinary permutation σ.

Consider two copies of the space V and denote them by V_q (coordinates) and V_p (impulses). Regard $\tilde{V} = V_q \oplus V_p$ to be an S-analogue of phase space. Define a symmetry S on $V^{\otimes 2}$ as follows:

$$S(q_i \otimes q_j) = S_{ij}^{kl} q_k \otimes q_l, \qquad S(p_i \otimes p_j) = S_{ij}^{kl} p_k \otimes p_l,$$
$$S(p_i \otimes q_j) = S_{ij}^{kl} q_k \otimes p_l,$$

where $S = (S_{ij}^{kl})$ is the symmetry on the space $V^{\otimes 2}$.

Definition. Consider an algebra A equipped with the multiplication $\circ : A^{\otimes 2} \to A$ and the symmetry $S : A^{\otimes 2} \to A^{\otimes 2}$. A is S-commutative if $\circ S = S$ and $S \circ^{(1)} = \circ^{(2)} S^{(1)} S^{(2)}$. An operator $\{\ ,\ \}_S : A^{\otimes 2} \to A$ is an S-Poisson bracket if A is an S-Lie algebra with respect to $\{\ ,\ \}_S$ and the S-Leibnitz equality holds,

$$\{\ ,\ \}_S \circ^{(2)} = \circ \{\ ,\ \}_S^{(1)} (\mathrm{id} + S^{(2)}) .$$

If there is an involution (complex conjugation) in A, then we require that $\{\bar{f}, \bar{g}\}_S = ;\overline{\{f, g\}}_S$ holds.

Consider the algebra $A = A_+(V) \otimes \mathbb{C}$, which is obviously S-commutative. There exists a unique Poisson S-bracket in this algebra, which is defined on the generators in the following way:

$$\{q_i, q_j\}_S = \{p_i, p_j\}_S = 0, \qquad \{p_i, q_j\}_S = -g_{ij}, \qquad \{q_i, p_j\}_S = g_{ij} .$$

Consider the following real element of A as a Hamiltonian:

$$h = g^{ij} p_i p_j + \omega^2 g^{ij} q_i q_j .$$

This element is S-invariant. Operators Q_i and P_i are quantum analogues of the "S-classical observables" q_i and p_i. The operators Q_i and P_i commutate according to the same formulae as q_i and p_i. Define a Lie S-bracket:

$$[Q_k, Q_l]_S = [P_k, P_l]_S = 0, \qquad [Q_k, P_l]_S = i\hbar g_{kl} .$$

An operator A_f can be defined for each element $f \in A$ by an S-analogue of the quantization procedure due to Weyl. We need an operator analogue $H = A_h$ of the Hamiltonian h. It is

$$H = g^{kl} P_k P_l + \omega^2 g^{kl} Q_k Q_l .$$

Introduce creation and annihilation operators

$$a_k^+ = (\omega Q_k - iP_k)(2\omega\hbar)^{-1/2}, \qquad a_k = (\omega Q_k + iP_k)(2\omega\hbar)^{-1/2} .$$

The symmetry acts on these operators in the following way:

$$S(a_i \otimes a_j) = S_{ij}^{kl} a_k \otimes a_l, \qquad S(a_i^+ \otimes a_j^+) = S_{ij}^{kl} a_k^+ \otimes q_l^+ ,$$
$$S(a_i \otimes a_j^+) = S_{ij}^{kl} a_k^+ \otimes a_l .$$

The Lie S-bracket of the operators a_k^+ and a_k are

$$[a_k^+, a_l^+]_S = [a_k, a_l]_S = 0, \qquad [a_k, a_l^+]_S = g_{ij} .$$

Express the Hamiltonian H in terms of the operators a_k^+ and a_k,

$$H=\omega\hbar g^{ij}(a_i a_j^+ + a_i^+ a_j)=\omega\hbar g^{ij}(2a_i^+ a_j+g_{ij})$$
$$=\omega\hbar g^{ij}(2a_i a_j^+ - g_{ij}) .$$

Consider a quantum system constructed in the Fock S-space $\Phi=\Lambda_+(V_q)\otimes\mathbb{C}$. Define the actions of the operators a_i^+ and a_i in the space Φ as follows:

$$a_i^+ f \quad = q_i f, \qquad a_i q_j = g_{ij} ,$$
$$a_i(f\circ g)=a_i f\circ g+\circ a_i^{(1)} S(f\otimes g) ,$$

where $\circ$ is the multiplication in the algebra Φ. The latter equality means that the a_i are derivations of Φ.

Define a pairing $\langle\ ,\ \rangle : \Phi^{\otimes 2}\to\mathbb{C}$ on Φ as follows:

$$\langle f, g\rangle = \begin{cases} 0, & f\in\Phi^{(i)},\ g\in\Phi^{(j)},\ i\neq j, \\ \langle 1, 1\rangle=1, & \\ i!\langle\ ,\ \rangle\cdots\langle\ ,\ \rangle^{(i)}(f\otimes g), & i=j\geqslant 1. \end{cases}$$

The reader can easily prove that this pairing is non-degenerate, S-symmetric and S-invariant and that the operators a_i^+ are S-adjoint to a_i.

Therefore the operators P_i and Q_i are S-self-adjoint. Since the operator H is an S-symmetric expression on P_i and Q_i it is S-self-adjoint as well. The eigenvalues of this operator are

$$\lambda_N=2N\omega\hbar+\omega\hbar p, \quad N=0, 1, \ldots ,$$

where p is the rank of the symmetry S. The eigenspace corresponding to the eigenvalue λ_N coincides with $\Phi^{(N)}$. This follows from the fact that if $f\in\Phi(N)$ then $g^{ij}a_i^+ a_i f=Nf$.

The tensor $1\otimes 1$ is the vacuum state for the Hamiltonian H.

The statistical sum for H is of the form

$$\sum \dim \Phi^{(N)} e^{-\lambda_N/k_B T}=e^{-\omega\hbar p/k_B T}\mathscr{P}_+(t) ,$$

where T is a temperature, k_B the Boltzmann constant and t denotes $e^{-2\omega\hbar/k_B T}$.

Thus the statistical sum for the energy operator H of this S-quantum system is a rational [since $\mathscr{P}_+(t)=\mathscr{P}_-(-t)^{-1}$] function of $e^{-\omega\hbar/k_B T}$ and differs from the "classical" one.

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